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REVIEWS

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Ordinary Differential Equations Texts

Reviewed by **David A. Sánchez**

Once, long ago, our profession prided itself on writing slim, elegant undergraduate textbooks, especially for upper division courses, and I'm sure many of us keep them on our bookshelves as ready references and to assuage our nostalgia. We lost the calculus textbook corpulence battle long ago, and judging by the seven ordinary differential equations I was asked to review [**1**, **2**, **3**, **4**, **5**, **6**, **11**], we can unfurl the white flag again. The average treatment of ODEs covered 561 pages, and, as with a luxury car, you get a lot of accessories—solutions manual, computer projects workbooks, software packages, Maple or Mathematica workbooks or programming manuals (some free but mostly not), and even a World Wide Web page to communicate with the authors or other users. All this for $70 or more (not including accessories or dealer preparation costs) for a book to be used in what is normally a one-semester course for mathematics, science, and engineering students.

Before I can ask "What happened?" I must half-heartedly plead a *mea culpa*. Approximately ten years ago, Dick Allen, Tom Kyner, and I published a book [**10**], now long out of print, that was a forerunner to the books of today. The ODE section covered 580 pages, though in today's enlarged page size and reduced font size it would be under 400, but the skeletal structure that I see in almost all of today's books was there. The colossal growth in size is partially due to more graphics, four-color printing, student projects, and even wildlife photographs, but mainly because ordinary differential equations has been pushed aside for the magic topic, modeling. I will say more about this later.

I taught out of one of the books I reviewed; the course was largely for engineers, used Maple (sparingly in my case) and met three hours a week, but I had to add an extra hour just to discuss assigned problems. All the authors glibly pronounce that their books are designed for a two-quarter or one-semester course. This must be an in-house joke, because I found myself galloping slapdash through the book, jumping over sections to get to the essentials, then jumping back, and finally arriving at my favorite beautiful topic, phase plane analysis of two-dimensional linear and nonlinear systems, with only two class meetings left. I know what some of the students thought after forking out more than eighty dollars for the book and a Maple manual and probably covering less than 50% of the written material—and I'm a fast lecturer!

The obvious reply from some of you would be "If you had Maple available, you should have taught directly from the computer—there's no reason ever to touch pencil and paper." I have real problems with that response, but now we're getting into the big question of whether educational technology should be essential to

learning or should be to enhance it, which is far afield from the intent of this review. Therefore I will move on to commenting on some of the standard topics found in the books I examined.

First-order equations. The number of pages devoted to a general introduction to the subject, usually via first-order equations, was well over a hundred in almost all the books I reviewed. Sometimes there were anywhere from two to four chapters that could include modeling, numerical methods, existence and uniqueness theory, and stability and bifurcation; one book had a separate chapter on exponential growth. My impression is that first-order equations have become the platform for demonstrating the theory, quantitative and qualitative analysis, as well as modeling, simply because it is so very easy when the solution is in R^1, and we are blessed with great graphical and numerical schemes.

But I think we have overdone it, sometimes in the theory, where I found extensive discussion of existence and uniqueness, continuous dependence on parameters, mini-comparison theorems, etc., and more often in the modeling, where I found pages of two- and three-paragraph descriptions of applications plus appendices or suggested projects. We should remember that students are not quite clear on what the local existence and uniqueness theorem means—they have probably never seen such a theorem since their calculus problems were all of the Find or Solve type—and the whole notion of a solution, which may exist or not, which may unexpectedly blow up or not, which may be stable or not, may create a lingering haziness. I will discuss modeling later.

There follow some comments.

1. The linear equation is usually discussed in the traditional form $dx/dt + P(t)x = Q(t)$ rather than $dx/dt = a(t)x + b(t)$. The latter leads naturally to the representation $dx/dt = A(t)x + B(t)$ for systems and when $a(t) = a$ then the solution $\exp(at)$ is the analogue of $\exp(At)$. Why do we use the first form? Because we want to use *integrating factors*, and may admonish the student not to use, and certainly not to memorize (heaven forbid!) the very natural variation of parameters (V of P) formula, which goes right over to systems. Note that most applications fall out with the $a(t)x$ term on the right-hand side, and that many students mess up the integrating factor as a consequence.

2. Why doesn't anybody use judicious guessing—the method of undetermined coefficients—to solve $dx/dt = ax + b(t)$, where $b(t)$ is "nice"—a polynomial, exponential, sine or cosine? Using integrating factors or V of P will usually lead to tedious integration by parts, e.g., when $b(t) = t^3$, whereas judicious guessing is a simple algebraic operation of comparing coefficients of like terms. It's also a nice warm-up for the second-order case.

3. Homogeneous equations are a relic of the past and should be eliminated. The finding of integrating factors to create exactness is a completely unmotivated and mysterious art. Besides, where are the applications?

4. Special equations—Bernoulli, Riccati, Clairaut, etc.—should have the same fate, or be left as exercises. An introductory text should cover (in order) right-hand sides $g(t)$, $f(x)$, $f(x)g(t)$, then the linear cases $a(t)x$, $a(t)x + b(t)$, and $ax + b(t)$, remembering that it is not a course in integration techniques or trick substitutions that work only for special equations.

5. Why discuss first-order difference equations? I believe it is because some authors are itching to talk about the logistic difference equation, period doubling, cobwebbing, and then chaos. Consequently they introduce a topic that has no real place in an ODE text.

Second-order and higher linear equations. There is confusion here since some books use "higher order" to mean derivatives of order $n > 2$ while others mean $n \geq 2$ but spend most of their time on $n = 2$. Let us be honest: there is no real reason to discuss scalar linear equations of order greater than two, except possibly to give a few as problems. They have no physical meaning (except for a few special fourth-order ones), the examples must be rigged since most students nowadays cannot factor polynomials (and some have trouble with the quadratic formula), and the equations certainly don't deserve a separate chapter. Comments:

1. There seems to be an obsession with proving numerous mini-results about linearity, superposition, linear independence, Wronskians, particular solutions, annihilator rules, etc. The primary object is to arrive quickly at a description of a fundamental set of solutions, i.e., the basis of the solution space of the homogeneous equation. Some authors are enamored of formality, e.g., "Find the differential operator that annihilates . . . ." Once we find that a fundamental set of solutions exists, the Wronskian test applies if we are presented with or create another set of solutions as candidates. For the constant coefficient case the machinery of roots of characteristic polynomials can be introduced, and is simple when $n = 2$. This points out why we should stick to second-order equations or the 2×2 first-order system.

2. Few books use the fact that the solutions satisfying the initial conditions $x_k(t_0) = e_k$, $k = 1, 2, \ldots, n$, where the e_k are the unit basis vectors in R^n, form a fundamental set of solutions. This is a simple consequence of the linear independence of the e_k and the uniqueness of the solution of the initial value problem. For second-order equations this corresponds to the pair of initial conditions $x(t_0) = 1$, $dx/dt(t_0) = 0$, and $x(t_0) = 0$, $dx/dt(t_0) = 1$. Happily content that such a set of solutions exists we can go about the business of finding other sets of candidates.

3. For me, the verdict is still out on the method of annihilators, but I'm happier using judicious guessing and seeing some examples worked out, followed by a table of strategies to follow for various forcing terms. What often happens otherwise is an enshrinement of operator notation (one book uses $Op[y] = f$) and lots of confusing manipulation. It gets worse when this formalism is extended to linear systems and we are confronted with matrices and determinants whose entries are products of operators. But I must admit that if I see that I'm not going to have the time to cover the theory for $dx/dt = Ax$, A a 2×2 matrix, then using D operators and some linear algebra to get a second-order equation for one of the components is quick and avoids all the difficulties. I'll stick to judicious guessing for $dx/dt = Ax + f(t)$ when $f(t)$ is fairly simple.

4. Alas, the V of P formula for second-order equations must be presented in its full Wronskianite glory and you can be assured that few will ever use it. So it remains a source of exercises in techniques of integration. For second-order constant coefficient cases few authors show the convolution form of the V of P integral where the kernel is the solution of the homogeneous equation with initial conditions $x(0) = 0$, $dx/dt(0) = 1$. This leads to a nice discussion of Green's or weighting functions in the t-domain, independent of Laplace transforms, and one can now handle simple forcing terms without resorting to transform methods. Incidentally, I'm inclined to believe that Euler equations are introduced because they are one nonconstant coefficient set of equations that can be solved (via a trick!) and are a source of grungy V of P problems. Students mess these up because they miss the fine print where it mentions that the V of P formula applies only when the leading coefficient of the differential equation is 1. I think there

should be a deemphasizing of Euler equations, and they could easily be relegated to problems.

5. Finally, I would like to comment in this section on the approach used by Blanchard, Devaney, and Hall in their forthcoming book [5], which could be regarded by some as the "new wave." They deliberately convert all second-order equations to first-order systems in order to be able to take advantage of graphical packages and phase plane analysis. When we get to studying qualitative behavior this is a correct approach, but not from the start. Most of our ODE students will encounter second-order equations expressed in the traditional form in their physics and engineering courses, mainly as a result of Newton's laws. They should be able to recognize immediately (maybe after a quick back-of-the-envelope calculation) a harmonic oscillator, damped free vibrations, and resonance, and should be able to describe the solution without having to transform the equation into a first-order system. Their professors and future employers will expect this, and mathematics departments may increase their reputations for woolly thinking if they don't preserve this very practical capacity.

Numerical methods. Some books intersperse elementary numerical methods when first-order equations are discussed; others do a whole chapter of thirty to forty pages, but eventually they all get into far more detail than is needed. I found lengthy discussions of Taylor's method and Runge-Kutta for single equations and systems, and further topics often included error propagation, stiffness, multi-step methods, and numerical instability. Remember that the course is ordinary differential equations and not numerical analysis! This emphasis is surprising, considering that almost all ODE software packages include a very accurate numerical scheme such as RKF45. I think the student should be briefly introduced to Euler's method as a prototype, learn about the order of a method (maybe comparing to the Improved Euler Method), and then let the computer do the rest, basking in the radiance of the accuracy available from a fourth- or fifth-order scheme.

Laplace transform. I assume that this topic is introduced because engineering departments insist on it (in one book it really looked like a painful afterthought). I checked with some of my electrical engineering faculty acquaintances and they still expect their students to have the Laplace transform in their mathematical backpacks. The bête noire is partial fractions, and thankfully a few books do some simple exercises and then refer the students back to their calculus books for the gory details, overlooking the fact that the topic is being downplayed there. Maple has a full Laplace transform package and does all the dirty work, but it's a good idea to have the students do a few pencil and paper problems and know how to use a table of transforms. The transform solution approach to second-order equations with discontinuous forcing terms is a joy to use, in contrast to solving directly a sequence of initial value problems. Few books spend any time on Laplace transforms for systems where the algebraic calculations may be easier than using V of P. Almost no books point out that the exponential matrix $\exp(At)$ is simply $\mathscr{L}^{-1}\{(sI - A)^{-1}\}$; for 2×2 systems this is a snap to compute.

Series solutions. Remembering that the coverage of infinite series is being reduced in many calculus courses, one wonders why most books have extensive discussions of series solutions (the record was eighty-three pages). Pages plod on with countless shiftings of indices, analysis of poles and singular points, and all the arcane detail that gives ODEs such a terrible reputation. I took a whole semester course at the University of Michigan from Professor Rainville, entitled Intermedi-

ate Differential Equations, and nearly the only topic covered was the hypergeometric equation, including something called "Kummer's 24 Solutions." I wonder how I maintained my love for the subject. (Professor Rainville's *Elementary Differential Equations*, which was first published in 1949 has just appeared in an eighth edition [**9**]. It has only 530 pages.) The topic of infinite series should be downplayed or even eliminated, though I can hear the screams from the special functions crowd. But if the goal is to introduce students to the Bessel and Legendre functions, which might be useful for those students who do some partial differential equations, then there must be a better way to do it—at least a briefer one. Naively, one could just introduce them as solutions of certain special differential equations, do some graphs and discuss some properties and leave it at that. The challenge for future textbook writers is to cover the topic in relatively painless brevity.

Linear and nonlinear systems. My concern here is that many authors want to use this topic to give an introductory course in linear algebra. One book went so far as to discuss four-dimensional systems and demonstrated the calculation of 4×4 determinants using cofactors; another went overboard on Gaussian elimination. I have found that some simple explanations of matrix and vector operations and determinants in the 2×2 case are easily understood by students, and then one can proceed to the important matters. We should keep remembering that factoring polynomials is a lost art and that any 3×3 or higher examples must consequently be rigged, whereas in the 2×2 case there are no restrictions.

The discussions of the exponential matrix $\exp(At)$ tend to be excessive, some even going so far as an analysis of its canonical form, overlooking that using it to represent solutions is in part an artifice and usually a luxury. Most of the time we would be quite satisfied if we found any fundamental matrix $\Phi(t)$ with which to compute solutions and set up the V of P representation. Authors seem to forget that $\exp(At) = \Phi(t)\Phi(0)^{-1}$; for the 2×2 case this is an easy calculation for students and obviates the excessive discussions. Computing $\exp(At)$ using power series is another futile exercise ("Is this the way we do it?", "No", "Then why do it?"), and one simple example is enough.

All the authors do a comprehensive job discussing stability and phase plane analysis, with the power of current day graphics clearly evident, but some seemed to forget that their texts are introductory works and consequently a light touch is appropriate. This is especially relevant since the instructor has probably rushed through the book to get to this final or near final chapter and won't have the time to give a thorough discussion. Consequently, I think topics I found such as Liapunov functions, stable and unstable manifolds, the Poincaré-Bendixon and Hartman-Grobman theorems, and relaxation oscillations are out of place in a book intended as an introduction and not as a compendium. Certainly the student should see a limit cycle via an example or a computing exercise, but to go beyond that is overkill.

Modeling. It is very evident that the colossal size of the current ordinary differential equations books results from the subject of ODEs having become the playing field for modeling. I venture to say that if the excessive analysis of the logistic equation and the various population models—predator-prey, competitive, coexisting, harvested—in addition to related "projects" were removed, the size of most books would shrink considerably.

I think many of the population models are unmotivated, bearing no relation to the real world ("Let x be a population of prey, y a population of predators . . . "),

and are created so as to be able to showcase the magic of computer graphics. I did find one study of the New Zealand Opossum Plague, however. Models purporting to analyze warfare (would history have changed if Napolean had these at Waterloo?), or arms races, are the playground of terribly soft social science, and we give mathematics a bad name by presenting them as having any degree of predictive power.

Modeling within the mushy world of population biology may be a palliative to those of us uncomfortable with or tired of the laws of physics, but what will be the reaction of physics or engineering students when they find the first twenty pages of their textbook devoted to an analysis of the logistic equation? Let's not forget that such students get all the modeling they need in their physics and engineering courses, and what they want from us are the tools (analytic, numeric, graphical) to enhance their understanding.

Furthermore, we seem to be driven in our modeling to want to achieve the final Valhalla of chaos. With respect to this topic, presentations ranged from a short but elegant analysis of Lorenz's equation to lengthy presentations of bifurcation, period doubling, Poincaré maps, strange attractors, and even a description of Smale's horseshoe map. I believe we had better rethink this topic and decide what is its proper place in an introductory text, which should include both experiment and analysis but should not be a showplace for sophistication.

If we want to increase the modeling capacity of our mathematics majors, which is an excellent goal, a better idea, being used in some schools, would be a modeling course instead of loading up the ODE course. For such a course we could require a knowledge of subjects like ODEs, PDEs, linear algebra, probability and statistics, and maybe numerical analysis, and do some real in-depth analysis. I am also concerned that the present trend may eventually put off colleagues in the physical sciences and engineering, who may decide to teach their own ODE courses. That would be a real loss for us, both academically and financially.

Conclusion. The opinions in this review are founded on a deep love, over thirty years, of the subject of ordinary differential equations, on having written research papers and textbooks on the subject, and on having listened to some of the immortals and many of the mortals discuss the beauty and profundity of the subject in its most pure and applied portrayals. I know that the authors of the textbooks that I have looked at share that love, but I am very disturbed by whatever motivations and pressures have led to the creation of these colossal tomes. The love of the subject and its simple beauty is lost in the detail, the excess of multi-colored text and graphics, the sometime lack of elegance, and the unwillingness to make hard choices.

This obviously reflects pressures coming from the editors and publishers whose principal concern is adoptions and believe that authors must try to satisfy the personal whims of faculty on many campuses, especially large ones, and especially if those faculty are on textbook selection committees. Some of these faculty have their pet topics that must be included, others are computers or graphics zealots, and some have no real understanding of the subject. In addition, the conservatism of publishers results in copy-cat books. "If Wronsky and Groanwell is getting lots of adoptions then we better make sure our book covers the same stuff—then jazz up the computing supplements and slap a zingy cover on it. Maybe something with limit cycles and spotted owls." Further pressure comes from engineering departments, who have their own list of non-negotiable demands. I have no solution to this problem—it must come from our community—but I believe it can be resolved.

Just as I started this review I came across the recently published book by Bob O'Malley [7]. It is a slim, 240-page exposition at a slightly higher level than introductory undergraduate, but it captures both the analytic and qualitative foundations of the subject, with good examples and problems, and even covers series solutions (but not Laplace transforms). It should be possible to write such a book at the undergraduate level, then supplement it with some computing workbooks, *not* programming manuals (Polking's [8] is a good example), to take advantage of the power of the computer in describing direction fields, doing phase plane analysis, and crunching numbers. Anyone of my generation who has plotted direction fields by hand and wondered where in blazes is the solution, or ground out Runge-Kutta calculations on a primitive calculator, has to appreciate how much current technology enhances an introductory course.

But that technology should not be used to solve $dx/dt = 2x$ or $d^2x/dt^2 + x = 0$, or move the subject to an "Enter the equation via the following four-line syntax-ridden command and then push the return key" mentality, which could rapidly become drill and not understanding. The foundation of that understanding is composed of only a few essential analytic blocks, but I'm afraid the current size of introductory texts has buried them in an avalanche of qualitative amusements. Maybe the watchword should be "Whoa!" and not "Wow!"

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